

# Solution of the Dirac Equation for Scalar Potentials and its Implications in Atomic Physics\*

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The Dirac equation is solved for mixed vector-scalar potentials. The limits on the scalar coupling constant are set by precision atomic experiments. We obtain the maximal size of the scalar coupling constant  $\alpha' = 2.5 \cdot 10^{-8}$  per charge unit. Deviations from the Mott scattering are also determined.

## I. Introduction

In a recent publication Dosch, Jensen and Müller<sup>1</sup> have shown that the properties of the Dirac equation change dramatically when a scalar external  $\alpha'/r$  potential is coupled instead of a vector Coulomb potential. They found that Kleins's paradox is not present. This implies, that for an arbitrary large coupling constant for the scalar potential one can always find the bound states in the gap between  $+m_e c^2$  and  $-m_e c^2$ . It is known<sup>2,3</sup>, that this is not the case for the usual Coulomb potential, which is the fourth component of the electromagnetic vector potential and is coupled to the Dirac field by replacing  $\mathbf{p}$  by  $\mathbf{p} - (e/c)\mathbf{A}$  in the Dirac equation. Using realistic assumptions<sup>2,3</sup> one finds that for a coupling constant larger than  $170\alpha$  the  $1s$  state joins the lower energy continuum. The subsequent positron production can possibly be observed experimentally in the collision of very heavy ions<sup>3</sup>. It is interesting to note that the presence of a small scalar coupling does not prevent this "diving" of the bound levels into the negative energy continuum and therefore also not the positron autoionization. This can be shown by solving the Dirac equation with mixed coupling terms corresponding to external vector and scalar potentials. It is clear that an additional scalar potential effects also the results of atomic spectroscopy known with high precession. Indeed, using the hydrogen fine structure experiments we can set an upper limit on the magnitude of the scalar coupling constant  $\alpha'$ .

Another particle, which obeys the Dirac equation, is the muon which, because of its large mass moves usually in regions of very strong electric field within the atom. One can thus expect that eventually

existing scalar potentials will show up in muonic atoms. Indeed, recently some disagreement between theory (including finite size effects and vacuum polarization up to order  $\alpha(Z\alpha)^3$  and  $\alpha(Z\alpha)^2$ ) and experiments has been reported for outer transitions such as  $5g_{7/2} - 4f_{5/2}$  in  $^{208}\text{Pb}$  and other nuclei<sup>4,5</sup>. The experimental situation is, however, not yet quite clear as different results have been published recently<sup>6</sup>. It is suggestive to try to attribute a small part of these discrepancies to an additional scalar coupling of the muon to the nucleus. In this connection we have first studied different phenomenological potentials in order to fit these discrepancies and found that additional potentials of the type  $1/r^n$  with  $n \geq 3$  outside of the nucleus are usually able to explain such deviations. However the effects on the inner cascade, such as  $3d_{5/2} - 2p_{3/2} - 1s_{1/2}$  were severe, that we were forced give up these trials<sup>7</sup>.

Sundaresan and Watson<sup>5</sup> have proposed an 8 MeV scalar meson with  $\alpha_s = g_{s\mu\mu} = g_{sNN} = 6 \times 10^{-7}$  to account for these above mentioned discrepancy.

This paper is organized as follows: In the subsequent section we solve the Dirac equation for mixed scalar and vector (Coulomb) potential and discuss some limiting cases. In Section III we apply these results to electronic and muonic atoms and set limits on the scalar coupling constant. In the final Sect. V we summarize our results. In Sec. IV we calculate the equivalent of the Mott cross section in the case of the scalar potential.

## II. Dirac Equation Coupled to Scalar and Vector External Potential

The Dirac equation ( $\hbar = c = 1$ )

$$[\gamma \cdot (p - eA) - m] \Psi = 0 \quad (1)$$

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admits analytic solutions only for a few especially simple potentials<sup>8</sup>. Among these are the Coulomb-potential and constant electric and magnetic fields. In the case of a purely electrostatic potential, the Dirac Eq. (1) can be written

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta - (E - V)] \Psi = 0. \quad (2)$$

Dosch, Jensen and Müller<sup>1</sup> have pointed out, that (2) is not the only possible way to couple a potential to the Dirac field. Although Eq. (1) and therefore Eq. (2) is the proper way to describe the influence of electromagnetic potentials it may be wrong in the case of interactions of other origin. It was suggested to consider the equation

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta(m + V) - E] \Psi = 0. \quad (3)$$

We want to generalize this idea to an equation containing both couplings with different coupling constants:

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta(m + V_2) - (E - V_1)] \Psi = 0. \quad (4)$$

We show that this equation admits analytic solutions for any  $V_1, V_2 \sim 1/|r|$ .

### II a) The Radial Equations

With  $V_1$  and  $V_2$  being external spherically symmetrical potentials it is easy to see<sup>9</sup> that the operator

$$\hbar \hat{K} = \beta(\boldsymbol{\sigma} \hat{I} + 1) \quad (5)$$

commutes with the Hamiltonian of Equation (4). As usual  $K$  has the eigenvalues  $k = \pm 1, \pm 2, \dots$ . After separation of the angular coordinates and insertion of a standard representation of the  $\alpha$ - and  $\beta$ -matrices one is left with the two radial equations

$$\begin{aligned} (E - m - (V_1 - V_2)) G + \left( \frac{d}{dr} - \frac{k}{r} \right) F &= 0, \\ (E + m - (V_1 + V_2)) F - \left( \frac{d}{dr} + \frac{k}{r} \right) G &= 0 \end{aligned} \quad (6)$$

where 
$$\Psi(\mathbf{r}) = \frac{1}{r} \begin{pmatrix} F(r) \chi_k^\mu \\ i G(r) \chi_{-k}^\mu \end{pmatrix}.$$

Choosing

$$\begin{aligned} V_1 &= -\alpha/r, \\ V_2 &= -\alpha'/r \end{aligned}$$

as the two radial "Coulomb"-like potentials the radial equations read

$$\left[ E - m + \frac{1}{r} (\alpha - \alpha') \right] G + \left( \frac{d}{dr} - \frac{k}{r} \right) F = 0, \quad (7a)$$

$$\left[ E + m + \frac{1}{r} (\alpha + \alpha') \right] F - \left( \frac{d}{dr} + \frac{k}{r} \right) G = 0 \quad (7b)$$

### II b) The Energy Eigenvalues

To solve the system (7) we introduce the abbreviations

$$A = m + E, \quad B = m - E, \quad D = \sqrt{AB} \quad (8)$$

and the dimensionless variable

$$\varrho = r D. \quad (9)$$

The ansatz

$$F(r) = e^{-\varrho} f(\varrho), \quad G(r) = e^{-\varrho} g(\varrho) \quad (10)$$

leads to the eqs.

$$\begin{aligned} \left[ B/D - \frac{1}{\varrho} (\alpha - \alpha') \right] g - \left[ \frac{d}{d\varrho} - \frac{k}{\varrho} - 1 \right] f &= 0, \\ \left[ A/D + \frac{1}{\varrho} (\alpha + \alpha') \right] f - \left[ \frac{d}{d\varrho} + \frac{k}{\varrho} - 1 \right] g &= 0. \end{aligned} \quad (11)$$

With a power series expansions for  $f(\varrho)$  and  $g(\varrho)$

$$f = \sum_{r=0}^{\infty} a_r \varrho^{s+r}, \quad g = \sum_{r=0}^{\infty} b_r \varrho^{s+r} \quad (12)$$

and demanding that the coefficient of each power of  $\varrho$  vanish, we obtain

$$\begin{aligned} (\alpha - \alpha') b_0 + (s - k) a_0 &= 0, \\ -(s + k) b_0 + (\alpha + \alpha') a_0 &= 0, \end{aligned} \quad (13a)$$

$$\begin{aligned} (B/D) b_r - (\alpha - \alpha') b_{r+1} - (s + r + k + 1) a_{r+1} \\ + a_r = 0, \quad r > 0 \end{aligned} \quad (13b)$$

$$\begin{aligned} b_r - (s + r + k + 1) b_{r+1} + (\alpha + \alpha') a_{r+1} \\ + (A/D) a_r = 0. \end{aligned} \quad (13c)$$

System (13a) is nontrivially soluble, if

$$\begin{vmatrix} \alpha - \alpha' & s - k \\ -s - k & \alpha + \alpha' \end{vmatrix} = s^2 - k^2 + \alpha^2 - \alpha'^2 = 0.$$

We thus obtain

$$s^2 = k^2 - (\alpha^2 - \alpha'^2). \quad (14)$$

Multiplying Eq. (13b) by  $D$  and Eq. (13c) by  $B$  and subtracting the first equation from the second one, we obtain

$$\begin{aligned} [B(\alpha + \alpha') + D(s + r - k)] a_r + [D(\alpha - \alpha') \\ - B(s + r + k)] b_r = 0 \end{aligned} \quad (15)$$

for every  $r > 0$ .

To obtain normalizable wave functions we have to require that the series (12) are polynomials, i. e. that  $b_r, a_r = 0$  for  $r > N$ . We obtain from Eq. (13b, c)

$$B b_N + D a_N = 0. \quad (16)$$

Let us denote Eq. (15) in the particular case  $\nu = N: {}^{+m}$

$$[D(\alpha - \alpha') - B(s + N + k)] + [B(\alpha + \alpha') + D(s + N - k)]a_N = 0. \quad (17)$$

Equations (16) and (17) again have nontrivial solutions  $a_N, b_N$  if

$$\begin{vmatrix} B(\alpha + \alpha') + D(s + N - k) & D(\alpha - \alpha') - B(s + N + k) \\ D & B \end{vmatrix} = (D^2 - B^2)\alpha - (D^2 + B^2)\alpha' - 2BD(s + N) = 0. \quad (18)$$

Resubstitution of Eqs. (8) for  $B$  and  $C$  finally leaves us with a condition on the energy eigenvalues

$$E\alpha + m\alpha' = (s + N)\sqrt{m^2 - E^2}. \quad (19)$$

A short calculation shows that

$$E = m(qq' \pm \sqrt{1 + (q^2 - q'^2)}) / (1 + q^2) \quad (20)$$

where we have used the abbreviations

$$q = \alpha / (s + N), \quad q' = \alpha' / (s + N). \quad (21)$$

The  $(\pm)$ -sign in front of the square root in (20) needs some additional consideration. It has to be chosen such that in Eq. (19)

$$E\alpha - m\alpha' \geq 0$$

or since  $m > 0$ , such that the inequality

$$\pm q\sqrt{1 + (q^2 - q'^2)} \geq q' \quad (22)$$

is fulfilled. There may be two different solutions when  $\alpha' < 0$ !

### II c) Cases of Interest

The three particular cases  $\alpha = 0$ ;  $-\alpha = \alpha'$ ;  $|\alpha'| = \varepsilon \ll \alpha$  are considered

1)  $\alpha = 0, \alpha' < 0$ .

This is the case studied by Jensen et al.<sup>1</sup>. The energy formula

$$E = \pm m [1 - \alpha'^2 / (N + \sqrt{k^2 + \alpha'^2})^2]^{1/2} \quad (23)$$

is shown in Figure 1. We see that for arbitrary large  $\alpha'$  the resulting eigenvalues tend asymptotically to  $E = 0$ . The expansion for  $|\alpha'| \ll 1$  and positive branch gives

$$E - m \approx -\frac{m\alpha'^2}{2n^2} \left[ 1 + \frac{\alpha'^2}{n} \left( \frac{1}{|k|} + \frac{1}{4n} \right) \right] \quad (24)$$

where

$$n = N + |k|. \quad (25)$$

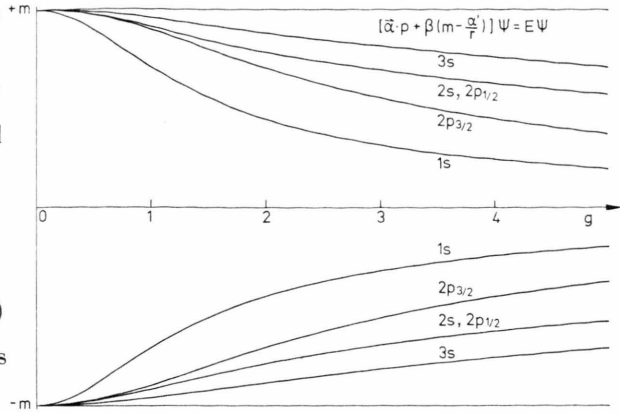


Fig. 1. The solutions of the Dirac equation with a purely scalar potential as function of the scalar coupling constant  $\alpha' = g$ .

For the usual Coulomb potential the fine structure formula reads

$$E_{\text{coul}} - m \approx -\frac{m\alpha^2}{2n^2} \left[ 1 + \frac{\alpha^2}{n} \left( \frac{1}{|k|} - \frac{3}{4n} \right) \right] \quad (26)$$

There is obviously a clear difference as far as angular dependence is concerned between both interactions even for small coupling constant.

2)  $\frac{1}{2}\alpha = -\frac{1}{2}\alpha'$ :

This mixed potential leads to the two energy spectra

$$E_1 = m [1 - \frac{1}{4}(\alpha/(s + N))^2] / [1 + \frac{1}{4}(\alpha/(s + N))^2], \quad (27a)$$

$$E_2 = -m. \quad (27b)$$

The second spectrum is a trivial solution to the wave equation, since

$$V = -\frac{1}{2} \frac{\alpha}{r} + \frac{1}{2} \beta \frac{\alpha}{r} = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha/r \end{pmatrix} \quad (28)$$

decouples the large from the small components as far as the potential term is concerned.

For  $\alpha^2 = \alpha'^2$  we have  $s = |k|$  [Eq. (14)] and the first energy spectrum becomes ( $n = N + |k|$ )

$$E = m \left[ 1 - \frac{\frac{1}{2}(\alpha/n)^2}{1 + \frac{1}{4}(\alpha/n)^2} \right]. \quad (29)$$

The energies given by (29) are especially interesting because they do not depend on the angular momentum number  $|k|$ . As can be seen from Fig. 2, in the limit  $\alpha' \rightarrow \infty$  the energies approach the negative energy continuum.

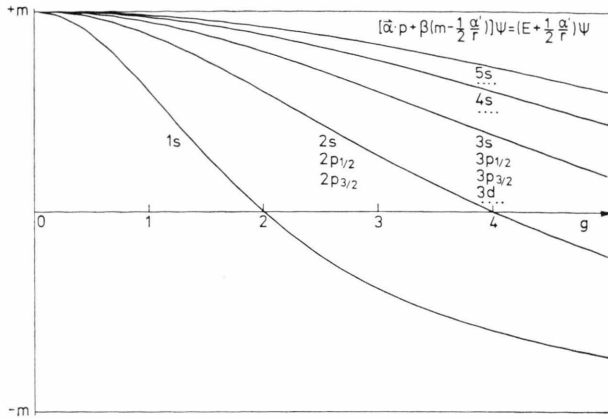


Fig. 2. The solutions of the Dirac equation with mixed scalar and vector potential for the case  $\alpha = -\alpha' = g$ .

3)  $|\alpha'| = \varepsilon \ll \alpha$ :

In this case Eq. (20) reduces to

$$E = m \frac{\alpha \alpha' + (s+N)^2 \sqrt{1 + \alpha'^2 / (s+N)^2}}{(s+N)^2 + \alpha^2}. \quad (30)$$

The above equation could be calculated in perturbation theory using the Coulomb wave function  $\Psi_{N,R}(\mathbf{r})$ . In passing we observe that a small scalar interaction in addition to the Coulomb potential will not change considerably the value of  $\alpha(Z_{\text{crit}})$  for which the 1s-state joins the negative energy continuum. This is noteworthy in connection with the problems of q.e.d. of strong fields discussed in Reference 3.

#### II d) The Radial Wave Functions

To complete our discussion of the Dirac equation coupled to scalar and vector potentials we consider now the radial wave functions. In order to obtain simple recurrence formula for the coefficients  $a_\nu$  and  $b_\nu$  in the expansion (12) we introduce  $c_\nu$  and  $d_\nu$  by

$$\begin{aligned} a_\nu &= \sqrt{B}(c_\nu + d_\nu), \\ b_\nu &= \sqrt{A}(c_\nu - d_\nu). \end{aligned} \quad (31)$$

Equation (15) transforms into

$$d_\nu = c_\nu \frac{B^2(\alpha + \alpha') + D^2(\alpha - \alpha') - 2BDk}{2BD(N - \nu)} \quad (32)$$

by using condition (18) to simplify the denominator.

Intertion of Eq. (32) into (13 b, c) where  $a_\nu$  and  $b_\nu$  have been substituted by  $c_\nu$  and  $d_\nu$  gives:

$$\begin{aligned} c_\nu &= -2 \frac{N - \nu}{\nu(2s + \nu)} c_{\nu-1}, \\ d_\nu &= -2 \frac{N - \nu + 1}{\nu(2s + \nu)} d_{\nu-1}. \end{aligned} \quad (33)$$

$c_0$  and  $d_0$  are connected by the relation (32):

$$d_0 = c_0 (\sqrt{k^2 + N^2 + 2sN - k})/N. \quad (34)$$

The recursion formulae (33) indicate that the solutions can be written in terms of confluent hypergeometric functions:

$$\begin{aligned} \sum c_\nu q^{s+\nu} &= c_0 q^s {}_1F_1(1 - N, 2s + 1, q/2), \\ \sum d_\nu q^{s+N} &= d_0 q^s {}_1F_1(-N, 2s + 1, q/2). \end{aligned} \quad (35)$$

Up to a constant normalization factor they are formally equivalent to the solutions of the electrostatic Coulomb problem (observe  $s = \sqrt{k^2 - (\alpha^2 - \alpha'^2)}$ !).

### III. Effects of Small Scalar Coupling in Electronic and Muonic Atoms

#### (a) Lambshift

The Lamb interval  $\mathcal{L} = \Delta E(2s_{1/2} - 2p_{1/2})$  is unaffected as can be seen from Eq. (30): The states  $2s_{1/2}$ ,  $2p_{1/2}$  are degenerate as in the usual Coulomb case.

#### (b) Inner transitions in heavy elements

Porter and Freedman<sup>10</sup> have measured inner transition energies in  $^{100}\text{Fm}$  with great precision. Theory<sup>11</sup> shows an agreement within  $\sim 30$  eV for the absolute energy of the 1s-level. Setting  $\Delta E = \pm 30$  eV in Eq. (30) we obtain

$$|\varepsilon_z| \lesssim 8 \cdot 10^{-5}.$$

Other transitions do not change this limit.

#### (c) Fine structure

The fine structure splitting  $\Delta E(2p_{3/2} - 2p_{1/2})$  of hydrogen is measured up to 0.04 MHz and is in agreement with theory<sup>12</sup>.

For small  $\alpha$  we obtain from Eq. (30) in addition to the usual fine structure:

$$E = \frac{m \varepsilon \alpha}{n^2} \left( 1 + \frac{\alpha^2}{n} \left( \frac{1}{|k|} - \frac{1}{n} \right) \right); \quad (36)$$

the contribution to the fine structure splitting is

$$\Delta E = \frac{m \varepsilon \alpha^3}{n^3} \left( \frac{1}{|k_1|} - \frac{1}{|k_2|} \right), \quad (37)$$

setting this contribution equal to 50 KHz we obtain

$$|\varepsilon_{fs}| \leq 1.6 \cdot 10^{-8}.$$

(d) *Inner muonic transitions*

Considering the detailed discussions of muonic X-rays in Thallium given by Backe, Engfer et al.<sup>13</sup> we can assume that in the energy of the 1s-state a contribution of about 3 KeV from an additional interaction could be absorbed by small variations of nuclear radius and by precise theoretical determination of nuclear polarisation. In the case of inner muonic transitions we should be more cautious when using Eq. (30): The finite size effect deforms considerably the muonic wave function. However, a rough estimate within a factor of 2 is possible, if one assumes an effective  $\alpha$  for Thallium of the order of  $(2/3) \cdot (Z/137)$ . Setting  $\alpha \approx 0.5$  and  $E = 3$  keV we obtain from Eq. (30) for  $\varepsilon_\mu$

$$|\varepsilon_\mu| \lesssim 6 \cdot 10^{-5}.$$

(e) *Outer muonic transition*

We discuss now transitions such as  $5g_{7/2} - 4f_{5/2}$ ,  $5g_{9/2} - 4f_{7/2}$  in lead and  $4f_{5/2} - 3d_{3/2}$ ,  $4f_{7/2} - 3d_{5/2}$  in Barium. The theoretical situation<sup>4</sup> is here much more clear than for inner transitions. There is almost no finite size effect present, the screening effect of electrons is still small (70 eV of 400 keV). The experiments in this region have been carried out in order to test vacuum polarization corrections,

which are dominant contributions to the Dirac binding energies in electrostatic potentials.

The experimental results, both of Dixit, Anderson et al.<sup>4</sup> and of Walter, Vuillumier et al.<sup>4</sup> show an accuracy of about 20 eV in the transition energies in the range of 200 – 440 keV. Deviations between theory and experiment as observed by Dixit, Anderson et al.<sup>4</sup> led to a recalculation of vacuum polarization and to new experiments. In Table I the final results of Refs. (4, 5) are given.

We have found, that a finite scalar coupling constant could explain all these deviations. Best agreement could be achieved by assuming that the strength of the coupling is proportional the number of nucleons in the nucleus. In Fig. 3 the results of such a fit are shown. The bars indicate the errors of the experiment. Obviously  $\alpha' = 0$  would be a bad choice. The best fit is obtained with the coupling constant  $\alpha' = 1.7 \times 10^{-7}$  per nucleon. This value for the nucleon-muon coupling constant corresponds in Tl to  $|\varepsilon| = 3.4 \times 10^{-5}$  and is in accordance with our limit for the inner transitions.

As this value is almost the limit we obtained from the finestructure experiments, it seems consistently possible to attribute a scalar charge to protons, electrons and muons, with the value of the coupling constant  $\alpha' = e_{\text{proton}} \times e_{\text{lepton}} \leq 2.5 \times 10^{-8}$ . The present experimental accuracy does not indicate any breaking of the electron-muon universality. We mention, that Rinker and Wilets<sup>14</sup> have considered this questile comparing electron scattering data with data obtained from muonic atoms with  $Z < 30$ .

Table 1. Comparison of theoretical and experimental transition energies in muonic atoms (Ref. <sup>4, 5</sup>). Theoretical values include vacuum polarization order  $\alpha(Z\alpha)$ ,  $\alpha^2(Z\alpha)$ ,  $\alpha(Z\alpha)^3$ ,  $\alpha(Z\alpha)^5$ , finite nuclear size, selfenergy, electron-screening, nuclear polarization, relativistic correction to reduced mass.

$Z$	Transition	$E_{\text{th}}$ (KeV)	$E_{\text{exp}}$ (KeV)	$E_{\text{th}} - E_{\text{exp}}$ (eV)
47Ag	$4f_{5/2} - 3d_{3/2}$	308.460	$308.428 \pm 0.019$	$32 \pm 19$
	$4f_{7/2} - 3d_{5/2}$	304.777	$304.759 \pm 0.017$	$18 \pm 17$
48Cd	$4f_{5/2} - 3d_{3/2}$	321.993	$321.973 \pm 0.018$	$20 \pm 18$
	$4f_{7/2} - 3d_{5/2}$	317.986	$317.977 \pm 0.017$	$9 \pm 17$
50Sn	$4f_{5/2} - 3d_{3/2}$	349.978	$349.953 \pm 0.020$	$25 \pm 20$
	$4f_{7/2} - 3d_{5/2}$	345.253	$345.226 \pm 0.018$	$27 \pm 18$
56Ba	$4f_{5/2} - 3d_{3/2}$	441.359	$441.299 \pm 0.021$	$60 \pm 21$
	$4f_{7/2} - 3d_{5/2}$	433.903	$433.829 \pm 0.019$	$73 \pm 19$
	$5g_{7/2} - 4f_{5/2}$	201.279	$201.260 \pm 0.016$	$19 \pm 16$
80Hg	$5g_{9/2} - 4f_{7/2}$	199.912	$199.902 \pm 0.015$	$10 \pm 15$
	$5g_{7/2} - 4f_{5/2}$	416.141	$416.087 \pm 0.023$	$54 \pm 23$
	$5g_{9/2} - 4f_{7/2}$	410.337	$410.284 \pm 0.024$	$53 \pm 24$
81Tl	$5g_{7/2} - 4f_{5/2}$	426.874	$426.828 \pm 0.025$	$46 \pm 25$
	$5g_{9/2} - 4f_{7/2}$	420.770	$420.717 \pm 0.023$	$53 \pm 23$
82Pb	$5g_{7/2} - 4f_{5/2}$	437.749	$437.687 \pm 0.020$	$62 \pm 20$
	$5g_{9/2} - 4f_{7/2}$	431.337	$431.285 \pm 0.017$	$52 \pm 17$

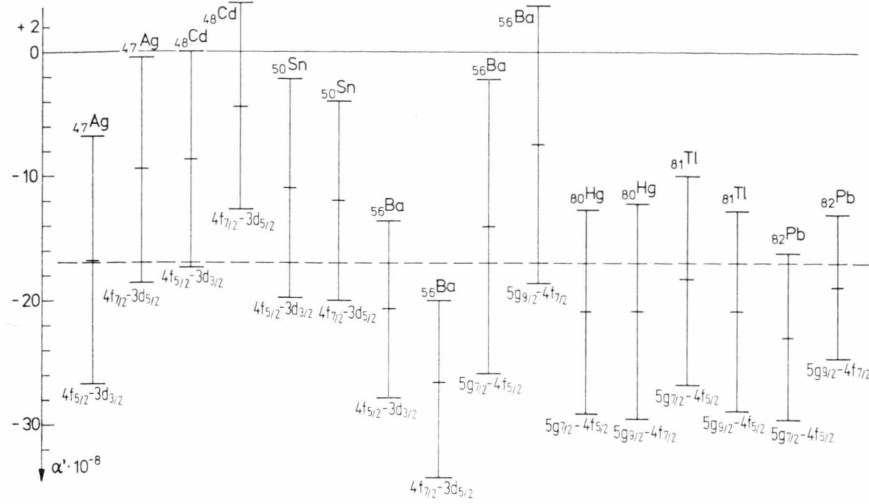


Fig. 3. Fit of scalar coupling constant from muonic data. The bars indicate the experimental errors. The horizontal line indicates the best average coupling constant  $\alpha' = 1.7 \cdot 10^{-7}$  per nucleon.

#### IV. Generalized Mott's Cross Section

It is not clear whether such a (effective) massless boson could be observed in scattering of electrons or muons on nuclei. The cross section will be dominated by usual electromagnetic interaction. Following Bjorken-Drell<sup>15</sup> we calculate in Born approximation the scattering of polarized muons on nuclei neglecting the recoil effects:

$$\begin{aligned} d\sigma/d\Omega &= (4m^2/|q|^4) \\ &\cdot \sum_{\pm s_f s_i} \{ \bar{U}(p_f, s_f) (\alpha \gamma_0 + \alpha') \sum (s_i) U(p_i, s_i) \} \\ &\cdot \{ U(p_i, s_i) (\alpha \gamma_0 + \alpha') U(p_f, s_f) \}. \end{aligned} \quad (38)$$

Here  $q$  is the momentum transfer and the  $U$ 's denote the spinors of the initial and final states with momentum  $p_i$  and  $p_f$  and spin projections  $s_i$  and  $s_f$ . The sums may be carried out, and we are left with

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{4m^2}{|q|^4} \text{tr}(\alpha \gamma_0 + \alpha') \\ &\cdot \left( \frac{1 + \gamma_5 \mathbf{s}_i}{2} \right) \left( \frac{\mathbf{p}_i + m}{2m} \right) (\alpha \gamma_0 + \alpha') \left( \frac{\mathbf{p}_f + m}{2m} \right). \end{aligned}$$

It is easy to see that the spin term  $\gamma_5 \mathbf{s}_i$  disappears after calculating the trace. Thus there is no special dependence on initial spin direction, when recoil effects are neglected. Therefore we consider:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{2m^2 \alpha^2}{|q|^4} \text{tr} \left( \gamma_0 \cdot \left( \frac{\mathbf{p}_i + m}{2m} \right) \gamma_0 \left( \frac{\mathbf{p}_f + m}{2m} \right) \right) \\ &+ \frac{2m^2 \alpha'^2}{|q|^4} \text{tr} \left( \left( \frac{\mathbf{p}_i + m}{2m} \right) \left( \frac{\mathbf{p}_f + m}{2m} \right) \right) \end{aligned}$$

$$\begin{aligned} &+ \frac{4m^2 \alpha \alpha'}{|q|^4} \text{tr} \left( \gamma_0 \left( \frac{\mathbf{p}_i + m}{2m} \right) \left( \frac{\mathbf{p}_f + m}{2m} \right) \right. \\ &\left. + \left( \frac{\mathbf{p}_i + m}{2m} \right) \gamma_0 \left( \frac{\mathbf{p}_f + m}{2m} \right) \right). \end{aligned} \quad (39)$$

The first term in (39) is the usual Mott's cross section ( $\alpha \rightarrow Z \alpha$ ):

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2}{4\beta^2 \mathbf{p}^2 \sin^4(\Theta/2)} (1 - \beta^2 \sin^2(\Theta/2)) \quad (40)$$

The second term gives the cross section in purely scalar potential ( $\alpha' \rightarrow Z \alpha'$ ).

$$\frac{d\sigma_{\text{scalar}}}{d\Omega} = \frac{4(Z\alpha')^2 m_\mu^2}{|q|^4} \frac{1}{2} \frac{p_i p_f + m_\mu^2}{m_\mu^2}. \quad (41)$$

Using:  $q^2 = 4p^2 \sin^2 \Theta/2$ ,  $P_i \cdot P_f = E^2 - p^2 \cos \Theta$  in the lab system one obtains

$$\frac{d\sigma_{\text{scalar}}}{d\Omega} = \frac{(Z\alpha')^2}{4\beta^2 \mathbf{p}^2 \sin^4(\Theta/2)} (1 - \beta^2 \cos^2(\Theta/2)). \quad (42)$$

This term is too small to be observed, because  $(\alpha'/\alpha)^2 = 1.2 \cdot 10^{-11}$ .

The third term in Eq. (39) is the interference term. One has

$$\begin{aligned} \frac{d\sigma_{\text{inter}}}{d\Omega} &= \frac{4}{|q|^4} \alpha \alpha' 2m(E_i + E_f) \\ &= \frac{\alpha \alpha'}{4\mathbf{p}^2 \beta^2 \sin^4(\Theta/2)} \sqrt{1 - \beta^2} \end{aligned} \quad (43)$$



and thus for Eq. (39) ( $a \rightarrow Z a$ ,  $a' \rightarrow Z a'$ )

$$\frac{d\sigma_{\text{total}}}{d\Omega} \approx \frac{(Z a)^2}{4 \mathbf{p}^2 \beta^2 \sin^4(\Theta/2)} \cdot \left( 1 - \beta^2 \sin^2(\Theta/2) + \frac{2 a'}{a} \sqrt{1 - \beta^2} \right). \quad (44)$$

The coefficient  $Z a'/a$  is equal to  $6.8 \cdot 10^{-6}$ . This means that for low energies (small  $\beta^2$ ) this interference effect would increase ( $a'$  is positive) Mott's cross section by about 0.0007%. This indicates that investigations of the effects of a scalar potential in electron and muon scattering is very difficult. To our knowledge, such precision is beyond present experimental possibilities. It should be noted that for  $a = \pm a'$  we can write the total interaction scattering cross section as

$$\left. \frac{d\sigma_{\text{total}}}{d\Omega} \right|_{a=\pm a'} = \frac{a^2}{4 \mathbf{p}^2 \beta^2 \sin^4(\Theta/2)} (2 - \beta^2 \pm 2 \sqrt{1 - \beta^2}) \\ = \frac{a^2}{4 \mathbf{p}^2 \beta^2 \sin^4(\Theta/2)} (1 \pm \sqrt{1 - \beta^2})^2.$$

There is no additional angular dependence of the cross section, only the Rutherford's  $\sin^4 \Theta/2$  is present. This is in accordance with our result that the discrete spectrum in this case is not dependent on angular quantum numbers.

## V. Conclusion

The presence of an effective scalar  $1/r$  potential, which acts mainly between leptons and protons, is compatible with the present day experimental situation. The agreement between theory and experiment can be even improved. Such potentials may be mediated by a massless scalar meson which couples to leptons and protons, the product of both coupling constants being, as determined,  $< 2.5 \cdot 10^{-8}$ .

On the other hand, one could think about an effective potential belonging to a pion-two photon

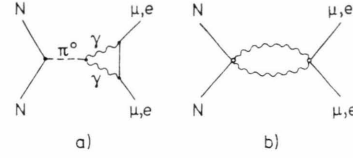


Fig. 4. a) A possible additional interaction of muons (electron) with nuclei. b) The limit when  $\pi^0$  and the intermediate fermion ( $\mu, e$ ) shrink to a point. The effective coupling constant depends on the mass of the fermion.

interaction as shown in Fig. 4 a which may be asymptotically described by two photon exchange graphs indicated in Figure 4 b. The two exchanged massless photons would simulate an exchange of a scalar particle. However, this effective potential would behave asymptotically like  $1/r^N$ ,  $N \geq 4$  on rather general reasons<sup>16</sup>. It has been shown<sup>7</sup> that such potentials cannot be introduced consistently into the muonic data.

We have shown that the half-half mixed scalar vector potential leads to eigenenergies of the Dirac equation, which depend only on the principal quantum number  $n$  and approach  $-m$  for arbitrary large coupling constant. As in this case the binding energy is as large as  $2m$ . It seems to be necessary to use the Bethe-Salpeter equation for a more realistic discussion.

Finally we draw attention to another interesting fact, namely that bound states are formed in a harmonic oscillator potential coupled to the mass term in contrast to pure vector coupled oscillator potentials, for which no bound states are possible. In the nonrelativistic limit the pure scalar oscillator reduces to the Schrödinger oscillator with biquadratic terms. It appears to us that this fact has been unknown up to now<sup>17</sup>.

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